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Stokes' Theorem: A candidate threshold concept.

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Abstract

An investigation into whether Stokes' Theorem, a mathematical concept in the field of Vector Calculus, is a threshold concept was conducted. This concept is taught as part of a second year undergraduate course for physicists at the University of Bristol. The investigation has two phases; first a theoretical examination of some of the criteria of a threshold concept to see if Stokes' theorem fits them, and then a student survey to analyse the applicability of the threshold concept label from the more subjective point of view of the learner. It was found that there is compelling evidence to support the case that Stokes' Theorem is indeed a threshold concept, and a brief discussion is made to assess how this identification can inform teaching and learning methods.

Keywords: Threshold Concept, Stokes' Theorem, Vector Calculus

Background

A threshold concept (TC) is a name for those concepts which, when understood by a student, mark a fundamental milestone in their learning. In demarcating the boundary between one level of learning and another, these concepts are of great importance for those teaching a section of knowledge. This is because identification of any possible candidate TCs in the curriculum allows lecturers to be able to tailor their teaching to better fit the unique needs and challenges which these concepts bring about. Clear identification of such topics during teaching is also useful for students, altering their sense of achievement by having a clear and important target for their learning as well as preparing them for the unique challenges of TCs. This inquiry will investigate whether

Stokes' theorem, coming from the field of vector calculus, is such a concept. If so, some of the consequences of this on the teaching of the theorem are then considered.

This investigation will be done in two main ways. First there will be a theoretical investigation into whether Stokes' theorem meets the criteria for and definition of a TC. Second, an investigation into students' opinions on the theorem, and how they situate this concept in the landscape of their course on vector calculus, will help assess the suitability of giving Stokes' theorem the status of a TC.
Finally, there will be a small discussion of how teaching practice may usefully change if Stokes' theorem truly is a TC.

Focus of the enquiry

A TC is characterised by the following criteria. Such a concept should be,

1. Transformative - changing the way students view the topic (Meyer, Land, & Baillie, 2010)
2. Troublesome - challenging students with accompanied confusion resolution cycles which may be repeated (Meyer & Land, 2003)
3. Irreversible - so difficult to unlearn
4. Integrative - bringing together seemingly disparate parts of the course (Holloway, Alpay, & Bull, 2010)
5. Discursive - altering and expanding the vocabulary and language available to the student (Meyer & Land, 2005)
6. Liminal - marking the boundary between proficiency and mastery of topic, or the boundary between different fields in a topic (Meyer, Land, & Baillie, 2010)

Some of these criteria can be investigated theoretically. In particular, criteria 2, 4, 5 and to some extent 6 may be able to be justified by analysing the place of Stokes' theorem in the field, without reference to the students' viewpoint. One goal will be to attempt justify these labels of Stokes' theorem through a theoretical analysis.

The other criteria are more subjective, and require understanding of the student experience as learners. So, the second goal will be to survey student opinions on the topic, but also to do so in way that does not lead the student in any particular direction or bias their answers.

The final goal would be to outline how small changes in teaching practice can help aid learning of TCs in general, and Stokes' theorem in particular, based on the feedback of students.

Positioning the question in the field

The primary literature on this topic can be separated into two types. First are those papers which have worked on defining the criteria and definitions of a TC. The development of the idea of a TC, along with the contributing criteria, were first proposed by Meyer and Land, (Meyer & Land, 2003) (Meyer, Land, & Baillie, 2010) (Meyer & Land, 2005). These works were further solidified by Cousin, (Cousin, 2006). It is important to note that there is no one clear and objective definition of a TC, and the definition is expected to continue to evolve and become more refined in the future.
The second set of papers of relevance are those that attempt to find candidate TCs in different fields. The literature on this is vast, with many subjects being looked at through the lens of TCs.

This project is looking at the mathematical topic of Stokes' Theorem, which arises in vector calculus. It is believed that no work has been done specifically on this topic. However, work has been done on finding such concepts in mathematics and vector calculus in general. For example, Breen and O'Shea (O'Shea, 2016) worked on looking in general at the applicability of TCs to undergraduate mathematics, while Craig and Campbell (Campbell, 2013) looked specifically at the idea of troublesome concepts in vector calculus. Finally, Worsley, Bulmer and O'Brien (Worsley, Bulmer, & O'Brien, 2011) also proposed that certain topics, such as multiple integrals (which also play a role in Stokes' theorem) may be TCs.

Breen and O'Shea's work in particular is useful for understanding the consequences of treating Stokes' theorem as a TC, and the methodology of the other papers guides some of the methodology of this work. However, the unique question undertaken here, "Is Stokes' theorem a TC?", is a new addition to the field.

Methods

As mentioned, this work consists of two inter-related parts. First there will be a theoretical discussion of some of the features of Stokes' theorem in an attempt to justify the applicability of some of the criteria given above. This discussion will mainly focus on the criteria of TCs being Integrative, Troublesome, Discursive, and Liminal.
The second part will focus on justifying the applicability of the other criteria through an understanding of the students experience as learners of this topic. This was done through a small questionnaire that was given to the students towards the end of the course. The course in which Stokes' theorem is taught is a second year undergraduate course on vector calculus that lasts for four weeks. Every other week of the course there is a problems class, where students attempt certain mathematical problems for a while before being shown the worked solutions. Since Stokes' theorem is the last concept to be taught, it forms the focus of the last problems class.

The cohort for the year, which is around one hundred and fifty students, are split into the thirds for the problem classes. This allowed for a variation in the questions given to the different groups. The questions which were common to all groups were:

1) Which topic in this course did you find the most challenging?
    a) Probability distributions
    b) Gradient Theorem
    c) Surface Integrals
    d) Volume Integrals
    e) Divergence Theorem
    f) Stokes' Theorem
    g) Other

2) What made this topic the most challenging?

3) Which topic required the greatest understanding of the rest of the unit to properly master?
    a) Probability distributions
    b) Gradient Theorem
    c) Surface Integrals

d) Volume Integrals
e) Divergence Theorem
f) Stokes' Theorem
g) Other

4) Which teaching methods did you find most useful in learning the topic you thought was most challenging?
   a) Lectures
   b) Online notes
   c) Practice Problems
   d) Problems class problems, attempts before the class
   e) Problem classes, during the class itself
   f) Online forums
   g) Talking to the lecturer
   h) Online videos from the lecturer
   i) Textbooks
   j) Other resources you found yourself

5) Which specific teaching method would you improve, and how, to help you understand the topic you found most challenging?

An extra question, specifically about TCs was asked to two of the three groups. After giving the students the definition of a TC as described above and talking them through the definition for around five minutes, the following additional question was asked:

6) Which topic, if any, was a threshold concept?

This question was not asked to all, since giving the definition of a TC could have caused some students to answer the first five questions differently from how they would have answered otherwise. This effect was not observed, however the precaution seemed sensible. Overall, 98 student responses were recorded, though not all answered every question. Finally, students were interviewed informally in small groups. This freeform discussion is certainly more subjective, and more open to the biases of the author influencing the student. However, despite this, certain thoughts and opinions expressed by the students were very valuable and pertinent to the question at hand.

Results and Analysis

Theoretical Discussion

In this first section, we will look at how far we can justify the application of the label of a TC purely theoretically. To begin this process, we first present what Stokes' theorem is. Stokes' theorem, in words, is the following result: If a vector field, $\underline{\boldsymbol{A}}$, can be written as the curl of another field, $\underline{\boldsymbol{F}}$, then the following relation will be true. That the flux of $\underline{\boldsymbol{A}}$ through some open surface is equal to the line integral of $\underline{\boldsymbol{F}}$ over the boundary of the surface. Mathematically, this is expressed as,

$$\iint_S \left(\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{F}}\right) \cdot \mathrm{d}\underline{\boldsymbol{S}} = \oint_C \underline{\boldsymbol{F}} \cdot \mathrm{d}\underline{\boldsymbol{r}}\,,$$

where *S* is some open surface, *C* is the boundary of *S*, and $\underline{\boldsymbol{F}}$ is some vector field.

Integrative Knowledge

For those not familiar with these mathematical terms, the key point is the following: Stokes' theorem requires the use of multiple other concepts. It involves: line integrals, surface integrals, the curl of a vector field and the flux of a vector field. In previous work in the field, (O'Shea, 2016) (Campbell, 2013), concepts like surface and line integrals on their own have been proposed as TCs. If those concepts are considered to be examples of Integrative knowledge, then by implication, Stokes' theorem would also have to be considered to have the same characteristic, since it builds upon those same foundations.

The use of these other concepts is not peripheral to the use of Stokes' theorem either; it is very much central. The key point about the theorem is that it is linking these concepts up and stitching them together to make a new whole which is greater than the sum of its parts. There simply is no way to understand Stokes' theorem without having a firm grasp of these multiple threads. Thus, based on this compound nature of the theorem, it seems clear that the theorem is a strong candidate to be considered as an example of Integrative knowledge.

Troublesome Knowledge

Not only does the theorem involve lots of other concepts, it also combines the concepts in a non-intuitive way, primarily since it involves the flux of the curl of a vector field. The flux of a vector field is a concept which is new to students of this course, as is the curl of a vector field, yet both these concepts have physical interpretations. The flux of a curl of vector field, however, defies any such easy interpretation and as such is a real challenge to many students.

The flux of field can be thought of in terms of fluid flow. Suppose your vector field describes the velocity of some fluid through space. The flux of the field through a surface in that space is then interpreted to be the amount of the fluid which is passing through the surface per unit time.

The curl of a field is a little harder to physically interpret but it is still possible. Again, we think of the vector field as describing the flow of some fluid in space. The curl of field calculates the rotational effect on the field on an infinitesimal sphere located at some point. This links to students' interpretation of angular momentum. But what is the flux of a curl of field? Here the fluid analogy breaks down and there is no easy physical interpretation, and without this to hold on to students find that they begin to struggle.

The other reason Stokes' theorem represents troublesome knowledge is that, due to the flux of the curl being non-intuitive, students often simply calculate the flux of a field instead and think this is the correct thing to do. They make this mistake presumably because they are familiar with the idea of the flux, and they know how to use it. In the informal discussions that were had with students, several students who had a record of doing very well in the problems class made this mistake. When it was pointed out to them that they had not applied the theorem correctly, and they needed to go back and really make sure they were doing it right, none of them could spot their mistake. Repeated hints and gestures as to where to look for their error were not helpful in moving the students on in their thinking, and only when the error was spelt out explicitly, could these students see what they had done. These were students who, when asked prior to attempting this question if they understood Stokes' theorem, all responded that they felt they had a good grasp. Their exasperation when faced with their error, and being unable to find the correct route even with hints, certainly dented their previous security in their knowledge. This example highlights that the back and forth cycles of understanding and confusion which accompany troublesome knowledge is indeed present in Stokes' theorem.

Discursive Knowledge

The most obvious change in the vocabulary of those that have mastered the theorem versus those who have not comes from understanding and being comfortable with the idea of the flux of the curl of a field. As mentioned before, this idea has no easy physical interpretation, yet being able to talk about mathematical problems in this way, opens up new methods for solving problems that cannot be attempted in any other way.

Liminal Knowledge

Purely in terms of the course structure which the students face during their undergraduate programme, Stokes' theorem comes up again in two very important sections of their degree. The first is in the mathematics of electromagnetism (EM), which is one of the central pillars of any degree in physics, and a requirement for Institute of Physics accreditation of their degree. In EM, Stokes' theorem is used as a key tool in solving several problems. The other area is that of complex analysis, where, again, many of the techniques covered in Stokes' theorem find their application and extension (such as Green's theorem). The course on complex analysis simply could not be undertaken without a grasp of Stokes' theorem, or at the very least the elements which make it up. Thus, Stokes' theorem certainly plays a gateway role in the teaching of students, opening new levels of knowledge, and access to entire methods which are very commonly used in their degree relies on understanding this concept.

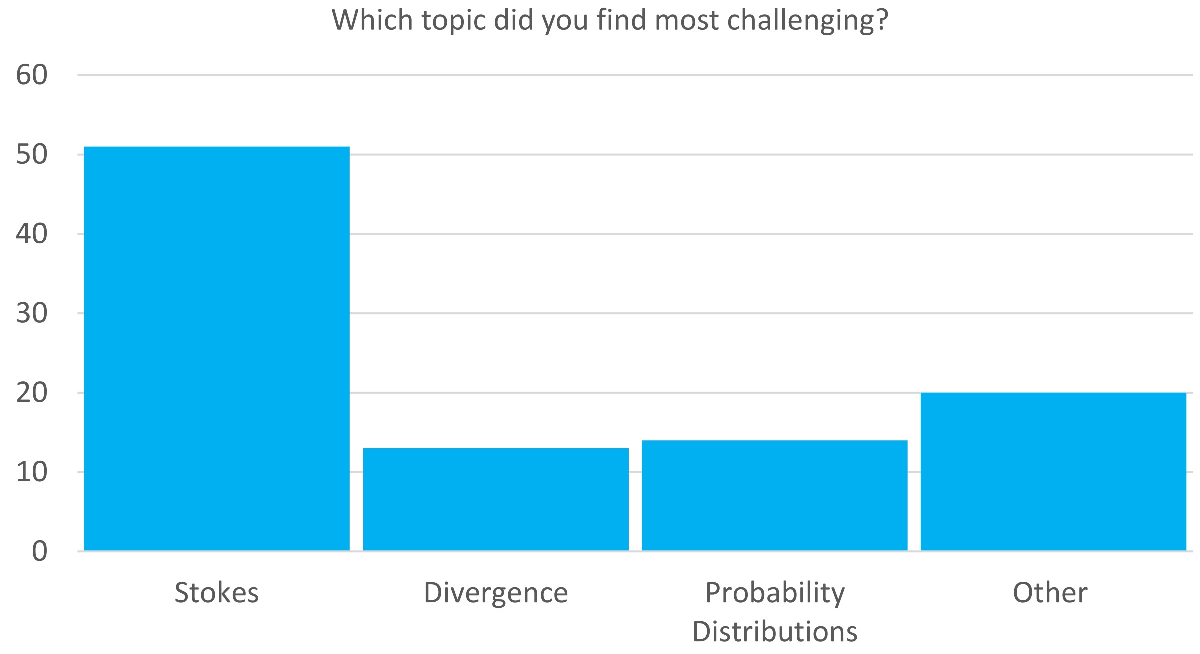
Student responses

<u>Which topic in this course did you find most challenging?</u>

The first question in the survey corresponds to criteria 2 of the definition of a TC. The results in Fig. 1 are quite clear: Stokes' theorem is considered to be the most challenging topic of the course. That proviso however, 'of the course', is important.

Simply because a topic is the most challenging, does not mean it qualifies as troublesome knowledge. However, in combination with the theoretical discussion about Stokes' theorem as troublesome above, this criterion does seem to be fulfilled by the theorem.

Figure 1. Results of question 1) from the survey. A total of 98 responses were collected.

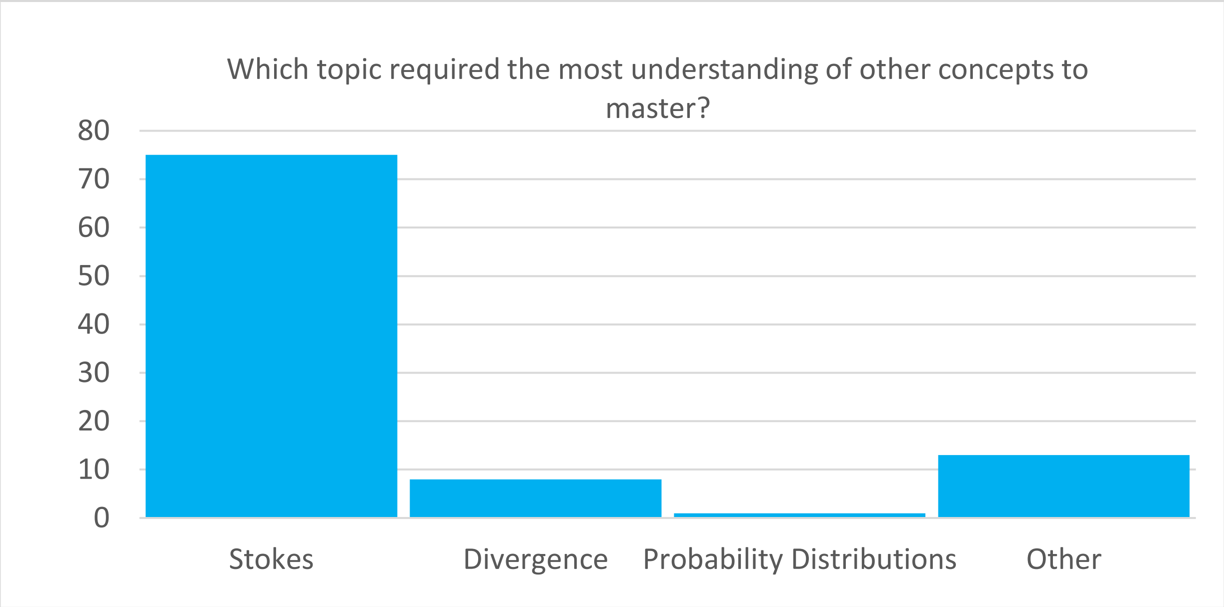


What made this topic the most challenging?

By far the most common response here was that Stokes' theorem was non-intuitive and lacked a physical analogy. This further backs up the theoretical discussion on troublesome knowledge above. One student noted that the theorem was 'key to understanding the rest of the course', since it required mastery of the topics and as such served as a benchmark for understanding. If you understood Stokes' theorem, then you understood the rest. In this sense, then, the criteria of transformative knowledge could also be applied. The next most common response was how hard it was to visualise the problem, which relates back to the lack of a direct physical analogy.

Going back to the informal student interviews, the group of students who at first thought they understood the theorem and then got stuck on a question also backs up the transformative nature of the theorem. Once the students were taught what they were doing wrong, there were several audible 'Oh!''s and 'Ah!''s around the room. Genuine understanding of the concept was repeatedly accompanied by a 'aha' moment for many students. The puzzle they were working on clicked into place and the picture it showed suddenly came into focus for them. Furthermore, it was observed that those students who had grasped this aspect required little to no further help on future questions. Ideally this would be investigated further, but this is some evidence to suggest that the understanding is irreversible

Figure 2. Results of question 3) from the survey. A total of 97 responses were collected.



<u>Which topic required the greatest understanding of the rest of the unit to properly master?</u>

This question relates to the Integrative nature of the concept, and the results as shown in Fig. 2 are even more clearly in favour of Stokes' theorem. There seems to be little double that Stokes' funnels several core concepts into one idea, both theoretically and in terms of the students' experience.

<u>Which teaching methods did you find most useful in learning the topic you thought was most challenging?</u>

This question does not relate directly to the main question of this work, but it sheds some light on how the teaching of Stokes' theorem may be changed to help the students. Students often chose more descriptive answers to this question, making a simple tally of results not possible. However, by far the most common response was that problem classes were the most useful way to learn troublesome concepts, and second to this was the online notes which students read in their own time.

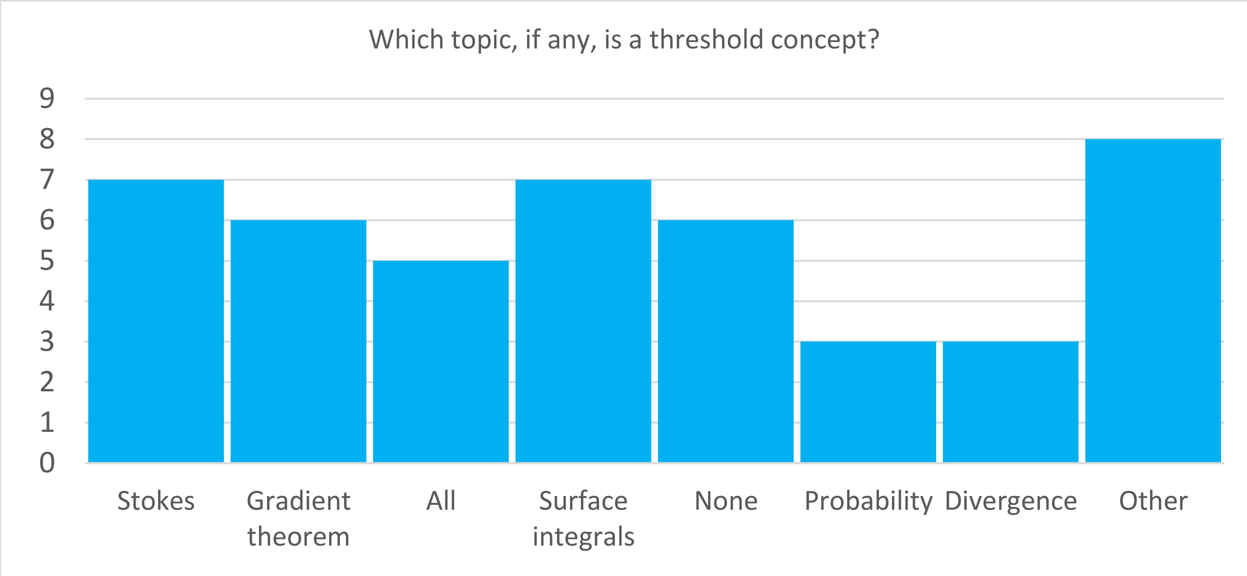
Problem based learning, as opposed to theoretical derivations of results, seems important precisely because of the tendency to trick oneself into thinking one understands a concept, and yet when faced with a problem in that field, not knowing what to do. The high response rate of this answer provides some evidence that Stokes' theorem is transformative, because it shows that simply understanding the components or even the derivation of the theorem (which is what the notes and lectures focus on) is not sufficient for mastery of the topic. Application to problems is essential to see the gaps in one's knowledge which are quite hidden otherwise.

<u>Which specific teaching method would you improve, and how, to help you understand the topic you found most challenging?</u>

Again, students were more descriptive in their responses here. However, the most common answers were more problems to work through, more problem classes, and more worked solutions. This is not surprising given the responses to the previous question, and the conclusions which can be drawn from it are much the same.
<u>Which topic, if any, was a threshold concept?</u>

Here the results, as shown in Fig. 3, were much more muddied. Student responses to this question varied a great deal, and it is especially noteworthy that five students actually thought that every concept listed was a TC. This, together with the range of responses, indicates that the discussion about the definition of a TC was not detailed enough. The discussion defining the idea took around five minutes, and it appears that this was not sufficient for students to fully get a grasp on what they were being told.

Figure 1. Results of question 6) from the survey. A total of 45 responses were collected



However, despite the range of responses, Stokes' theorem is jointly tied in first place as the single concept that most students thought could be a TC. So, although the conclusion is muddied by the range of other responses, this is certainly indicative that Stokes' theorem may be a good candidate for this label.

Discussion and conclusions

The results of the above analysis are quite interesting. By going through the definition of a TC and examining each criterion, evidence has been gathered that Stokes' theorem may well be a good candidate for such a concept. Theoretically it can be shown that the form of the theorem fits well with a concept that is Integrative, Troublesome, Discursive, and Liminal. Empirically students report that indeed their own experience of learning this topic means that they too would characterise the topic as being Troublesome, Transformative, Integrative.

The one criteria which was hardest to discern in Stokes' theorem was its Irreversibility. Since the students had just learnt the topic when the questionnaire was taken, it is understandable that they would not be in a good position to reflect on this aspect, and, as such, questions on this topic seemed ill suited to the questionnaire. There were hints at this criterion being involved in Stokes' theorem, as mentioned in the analysis section, however this is an area for future study. Ideally students who studied the theorem in their previous year could be surveyed and their opinion on the topic would greatly clarify this issue.

Lastly, despite some apparent difficulty in understanding the precise definition of TC, there was some indication that students too would indeed give Stokes' theorem this title. In conclusion, then, it appears that there is good reason to treat Stokes' theorem as such a concept when teaching this in the future. The reasons this would be a beneficial way of teaching, and a useful benchmark or milestone to highlight to the students is well explained in the past literature on using TCs to mark boundaries in student progress (Campbell, 2013) (Cousin, 2006) (Meyer & Land, 2003) (Meyer & Land, 2005) (Meyer, Land, & Baillie, 2010). The key points are that by highlighting to the students the special status of parts of the course, and how the many parts of the course are all tied into such concepts prepared students better for learning and mastering such topics. Being forewarned of their complexity allows for better preparation and anticipation of frustration can mitigate loss of motivation that sometimes accompanies troublesome knowledge. Finally, being aware of the liminal, transformative nature of the topic can itself motivate students to really get to grips with a topic in a way that can be lost in the general routine of university exam revision. All of this can only occur if students are told about the idea of TCs during the course, and as the data above shows, simply understanding the definition is itself a challenge.

So the main conclusion is that there is good evidence to suggest that Stokes' theorem is a TC, though more work is needed in this area, particularly on Irreversibility, and that, once identified as such, this can feed back into adaptations of teaching which motivate, orientate and prepare the student in very educationally useful ways.

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